BICHARACTERS, BRAIDS AND JACOBI IDENTITY

JERZY RÓŻAŃSKI

Abstract. For an abelian group G we consider braiding in a category of G-graded modules $\mathcal{M}^{kG}$ given by a bicharacter χ on G. For (G,χ)-bialgebra A in $\mathcal{M}^{kG}$ an analog of Lie bracket is defined. This bracket is determined by a linear map $E \in \mathrm{Lin}(A)$ and n-ary operations Ω_E^n on A. Our result states that if $E(1) = 0,\ E^2 = 0$ and $\Omega_E^3 = 0$ then a braided Jacobi identity holds and the linear map E is a braided derivation of a braided Lie algebra.

1. Introduction

For an abelian group G a monoidal category of G-graded modules $\mathcal{M}^{kG}$ has braiding $\mathcal{B}$ given by a bicharacter χ on the group G. In [6] Pareigis considered a theory of n-ary Lie algebras in a braided monoidal category for a model of braiding which is generated by a bicharacter. For some bicharacters Pareigis defined a n-ary bracket with two types of Jacobi identities. For a binary operations a braided Lie algebra was considered as a generalization of formal Lie theory by Gurevich for symmetrical braiding [2] and for pure braiding a braided Lie bialgebra was introduced by Majid [4].

The aim of this short letter is to study a relationship between braiding and Jacobi identity. Our viewpoint consists a characterization of this problem by some differential condition due to Koszul [3].

Consider (G,χ)-bialgebras [6] in the category $\mathcal{M}^{kG}$ which are $\mathcal{B}$-commutative. We start from the definition of n-ary operation Ω_E^n on a bialgebra for an arbitrary linear map E but restrict ourselves to some brackets defined by binary operations Ω_E^2 only. If $E(1) = 0,\ E^2 = 0$ and $\Omega_E^3 = 0$ then the Jacobi identity for the binary operation holds. In this case the linear map E is a braided derivation of a Lie bracket. Moreover we construct a braided derivation of an underlying algebra.

2. Notations, Definitions and remarks

Let $\Bbbk$ be a field and let $\Bbbk^*$ be the multiplicative group of $\Bbbk$. Denote by G an abelian group written additively, $g_i \in G,\ i = 1,\ 2,\ 3.$

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Definition 2.1 ([1])**.** *A function* $\chi : G \times G \to \mathbb{k}^*,\ \chi_{g_1,\,g_2} \in \mathbb{k}^*$ *is a bicharacter of* G *if*

$$\chi_{g_1+g_2,\,g_3} = \chi_{g_1,\,g_3}\chi_{g_2,\,g_3}, \qquad \chi_{g_1,\,g_2+g_3} = \chi_{g_1,\,g_2}\chi_{g_1,\,g_3}$$

A bicharacter is symmetric when $\chi_{g_1,\,g_2} = \chi^{-1}_{g_2,\,g_1}$. In this paper we will use nonsymmetrical bicharacters.

For objects $L,\ M \in \mathcal{M}^{kG}$ we have the tensor product

$$(L \otimes M)_g = \oplus_{h\in G} L_h \otimes M_{g-h}$$

and we have a braiding $\mathcal{B}$ for homogeneous elements $l \in L,\ m \in M$ with the corresponding degrees $|l|,\ |m|$,

$$\mathcal{B} : L \otimes M \ni l \otimes m \to \chi_{|l|,\,|m|}\ m \otimes l \in M \otimes L. \tag{2.1}$$

Note that $\mathcal{M}^{kG}$ with $\mathcal{B}$ is a strict braided monoidal category [1].

Consider $(G,\ \chi)$-algebras in the category $\mathcal{M}^{kG}$. An algebra is a pair $A = \{L,\ m\}$ where $L \in \mathcal{M}^{kG}$ and $m : L^{\otimes 2} \to L$, the multiplication in the algebra A preserves inclusions $L_g L_h \subset L_{g+h}$. For algebras $A,\ B$ the object $A \otimes B$ is again an algebra with a multiplication as composition

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathcal{B}} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

Above definition works similarly with a (co)multiplication for categories of (G, χ)-coalgebras, bialgebras and Hopf algebras.

In [5] we consider a braided derivations of an algebra. In this paper we have an interesting example of such derivations where a nonsymmetrical braiding is constructed by a bicharacter on a group.

Definition 2.2 ([5])**.** *A braided derivation of an associative algebra* $A = \{L,\ m\}$ *is* $\mathbb{k}$*-linear map* $d \in \mathrm{lin}_{\mathbb{k}}(L,\ L)$ *that satisfies the Leibniz rule*

$$d \circ m := m \circ (d \otimes \mathrm{id}_L) + m \circ B^{-1} \circ (d \otimes \mathrm{id}_L) \circ B$$

For consistent conditions of derivations with the associativity of an algebra A see in [5].

3. Main Propositions

Let $L \in \mathcal{M}^{kG}$ and (G, χ)-bialgebra $A = \{L,\ m,\ \triangle\}$ be $\mathcal{B}$-commutative. Let us fix the comultiplication $\triangle : A \to A^{\otimes 2}$ such that $\triangle(a) = a \otimes 1 - 1 \otimes a$. The extension of the comultiplication $\triangle$ is a map $\triangle^n : \otimes^n A \to A \otimes A$

$$\triangle^n(a_1 \otimes ... \otimes a_n) = \triangle(a_1) \cdot ... \cdot \triangle(a_n).$$

For example $\triangle(a)\triangle(b) = ab \otimes \mathrm{id} - a \otimes b - \chi_{a,\,b} b \otimes a + \mathrm{id} \otimes ab$.

For a linear map $E \in \mathrm{Lin}(A)$ one can define [3] the n-form $\Omega^n_E : A^{\otimes n} \to A$,

$$\Omega^n_E(a_1 \otimes ... \otimes a_n) = m \circ (E \otimes \mathrm{id})\triangle^n(a_1 \otimes ...a_n).. \tag{3.1}$$

For example

$$\Omega^2_E(a \otimes b) = E(ab) - E(a)b - \chi_{a,\,b}E(b)a + E(1)ab, \tag{3.2}$$

$$\Omega^3_E(a \otimes b \otimes c) = E(abc) - \chi_{b,\,c}E(ac)b - \chi_{a,\,b+c}E(bc)a \tag{3.3}$$
$$+\chi_{a+b,\,c}E(c)ab - E(ab)c + E(a)bc + \chi_{a,\,b}E(b)ac - E(1)abc.$$

Let us assume that a bicharacter satisfies

$$\chi_{a,\,b}\chi_{E(a),\,E(b)}\chi^{-1}_{a,\,E(b)}\chi^{-1}_{E(a),\,b} = -1. \tag{3.4}$$

Lemma 3.1. *For bicharacters (3.4) and $E(1) = 0$ we have*

$$\Omega^2_{E^2}(a \otimes b) = E\Omega^2_E(a \otimes b) + \Omega^2_E(E(a) \otimes b) + \chi_{a,\,b}\chi^{-1}_{a,\,E(b)}\Omega^2_E(a \otimes E(b)).$$

Proof. For the right side (3.2) with (3.4) is used. □

The three-ary operations Ω^3_E can be expressed by the binary Ω^2_E.

Lemma 3.2.

$$\Omega^3_E(a \otimes b \otimes c) = \Omega^2_E(a \otimes bc) - \Omega^2_E(a \otimes b)c - \chi_{b,\,c}\Omega^2_E(a \otimes c)b.$$

Proof. From (3.3) the right side has three terms

$$\Omega^2_E(a \otimes bc) = E(abc) - E(a)bc - \chi_{a,\,bc}E(bc)a + E(1)abc,$$
$$-\,\Omega^2_E(a \otimes b)c = -E(ab)c + E(a)bc + \chi_{a,\,b}E(b)ac - E(1)abc,$$
$$-\,\chi_{b,\,c}\Omega^2_E(a \otimes c)b = -\chi_{b,\,c}E(ac)b + \chi_{b,\,c}E(a)cb$$
$$+\,\chi_{b,\,c}\chi_{a,\,c}E(c)ab - \chi_{b,\,c}E(1)acb.$$

Due to the $\mathcal{B}$-commutativity the sum is $\Omega^3_E(a \otimes b \otimes c)$. □

Let us assume that a bicharacter satisfies

$$\chi_{a,\,b+E(bc)}\chi^{-1}_{a,\,E(b)+b+c} = 1, \quad \chi_{b,\,E(c)}\chi_{a,\,E(bc)}\chi^{-1}_{a+b,\,E(c)} = 1, \tag{3.5}$$
$$\chi_{a+b,\,c}\chi_{\Omega(a,b),\,E(c)}\chi^{-1}_{a+b,\,E(c)} = -1, \quad \chi_{a,\,b}\chi_{E(b),\,c}\chi_{\Omega^2_E(a,c),\,E(b)}\chi^{-1}_{a,\,E(b)}\chi^{-1}_{b,\,c} = -1.$$

Lemma 3.3. *For bicharacters (3.4), (3.5) and $E(1) = 0$ we have*

$$\Omega^2_E(\Omega^2_E(a\otimes b)\otimes c) + \chi_{b,\,c}\Omega^2_E(\Omega^2_E(a\otimes c)\otimes b) + \chi_{a,\,b+c}\chi^{-1}_{a,E(bc)}\Omega^2_E(a, \Omega^2_E(b\otimes c))$$
$$= \Omega^3_{E^2}(a \otimes b \otimes c) - E\Omega^3_E(a \otimes b \otimes c) - \Omega^3_E(E(a) \otimes b \otimes c)$$
$$-\,\chi_{a,\,b}\chi^{-1}_{a,\,E(b)}\Omega^3_E(a \otimes E(b) \otimes c) - \chi_{a+b,\,c}\chi^{-1}_{a+b),\,E(c)}\Omega^3_E(a \otimes b \otimes E(c)).$$

Proof. For the right side the lemma 3.2 for Ω^3_E with (3.5) is used. The corresponding terms we can write in the following way:

$$\begin{aligned}\Omega^3_{E^2}(a\otimes b\otimes c) &= E\Omega^2_E(a,bc)+\Omega^2_E(Ea,bc)+\chi_{a,bc}\chi^{-1}_{a,E(bc)}\Omega^2_E(a,E(bc))\\ &- E(\Omega^2_E(a,b))c-\Omega^2_E(Ea,b)c-\chi_{a,b}\chi^{-1}_{a,E(b)}c-\chi_{b,c}E(\Omega^2_E(a,c)b)\\ &- \chi_{b,c}\Omega^2_E(Ea,c)b-\chi_{b,c}\chi_{a,c}\chi^{-1}_{a,Ec}\Omega^2_E(a,Ec)b,\end{aligned}$$

and

$$\begin{aligned}E\Omega^3_E(a\otimes b\otimes c) &= E\Omega^2_E(a,bc)-E(\Omega^2_E(a,b)c)-\chi_{b,c}E(\Omega^2_E(a,c)b),\\ \Omega^3_E(Ea\otimes b\otimes c) &= \Omega^2_E(Ea,bc)-\Omega^2_E(Ea,b)c-\chi_{b,c}\Omega^2_E(Ea,c)b,\\ \Omega^3_E(a\otimes Eb\otimes c) &= \Omega^2_E(a,E(b)c)-\Omega^2_E(a,E(b))c-\chi_{E(b),c}\Omega^2_E(a,c)E(b),\\ \Omega^3_E(a\otimes b\otimes Ec) &= \Omega^2_E(a,bE(c))-\Omega^2_E(a,b)E(c)-\chi_{b,E(c)}\Omega^2_E(a,E(c))b. \qquad \square\end{aligned}$$

Denote by $D\in \mathrm{Lin}(A)$ the special linear map such that

$$\Omega^3_D=0,\qquad D^2=0\qquad \text{and}\qquad D(1)=0. \tag{3.6}$$

Proposition 3.4 (Jacobi identity)**.** *For the linear map (3.6) $D\in\mathrm{Lin}(A)$ and bicharacters 3.4 , 3.5 we have an identity for the binary operations*

$$\Omega^2_D(\Omega^2_D(a\otimes b)\otimes c)+\chi_{b,\,c}\Omega^2_D(\Omega^2_D(a\otimes c)\otimes b)+\chi_{a,\,b+c}\chi^{-1}_{a,D(bc)}\Omega^2_D(a,\Omega^2_D(b\otimes c))=0$$

Proof. The proof is based on the lemma 3.3 with (3.6). $\square$

From the binary operations Ω^2 we can construct a braided derivations of a multiplication of the algebra A by a term $d_a=\Omega^2_D(a,\cdot)$.

Proposition 3.5. *For the linear map (3.6) $D\in\mathrm{Lin}(A)$ we have*

$$\Omega^2_D(a\otimes bc)=\Omega^2_D(a\otimes b)c+\chi_{b,\,c}\chi^{-1}_{\Omega^2_D(a,\,c),\,b}b\Omega^2_D(a\otimes c).$$

Proof. This follows from the lemma 3.2

Consider a function $c: G\times G\to \Bbbk$, $\forall g_1,\ g_2\in G, c(g_1,\ g_2)\equiv c_{g_1,\,g_2}\in \Bbbk$. For the linear map D the bracket $[\cdot,\cdot]_D$ is defined via the binary operation Ω^2_E

$$c_{|a|,|b|}[a,\ b]_D := \Omega^2_D(a,\ b). \tag{3.7}$$

Let a function c satisfies

$$c(|a|,|b|)=-c(|D(a)|,|b|),\qquad c(|a|,|b|)=-c(|a|,|D(b)|)\in\Bbbk. \tag{3.8}$$

Proposition 3.6 (Derivation)**.** *For the bracket (3.7) with (3.8) the linear map (3.6) $D\in\mathrm{Lin}(A)$ is a braided derivation,*

$$D[a,\ b]_D=[D(a),\ b]_D+\chi_{a,\,b}\chi^{-1}_{a,\,D(b)}[a,\ D(b)]_D.$$

Proof. For the special linear map (3.6) the right side of the lemma 3.1 is equal to zero. □

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Department of Mathematical Methods in Physics, University of Warsaw, Hoża 74, PL-00-682 Warszawa, Poland.
E-mail address: rozanski@fuw.edu.pl